Financial Management
AFIN253

Lecture 1

Financial Maths and Debt Valuation

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Why is this subject important?

After finishing this subject, you will be able to:

- Calculate how much you can afford to borrow to buy a house.
- Estimate the price of a house, business, share, bond or any asset.
- Discuss rental and dividend yields, capital returns, total returns, risk, hedging and inflation with confidence.
- Avoid losing money in too-good-to-be-true ventures.
- Recognise the pitfalls of applying accounting concepts naively. Accountants often overlook opportunity cost and the time value of money.

- Understand the effects of debt (leverage), tax and negative gearing on returns and cashflows.
- Apply mathematics to real-world financial problems.
- Help your career and find employment in finance, business, accounting, real estate, management, sales and others.
- Have more friends after struggling through an interesting and very difficult subject.

Present Value of a Single Cash Flow

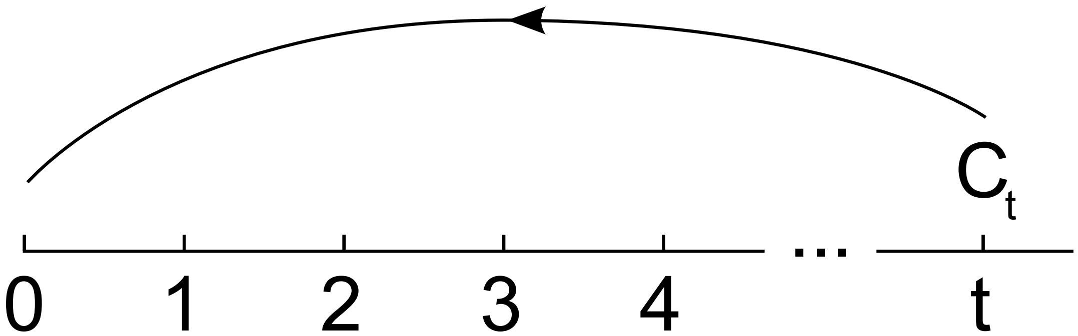
$$PV(single\ cash\ flow) = V_0 = \frac{C_t}{(1+r)^t}$$

Where:

C_t = cash flow at time t.

t = time periods.

r = the effective rate over a single period.



Calculation Example: Present Value of a Single Cash Flow

Question: What is the present value of $100 received in 5 years when interest rates are 8% pa?

Answer:

$$V_0 = \frac{C_t}{(1+r)^t}$$

$$= \frac{100}{(1+0.08)^5}$$

$$= 68.0583$$

Future Value of a Single Cash Flow

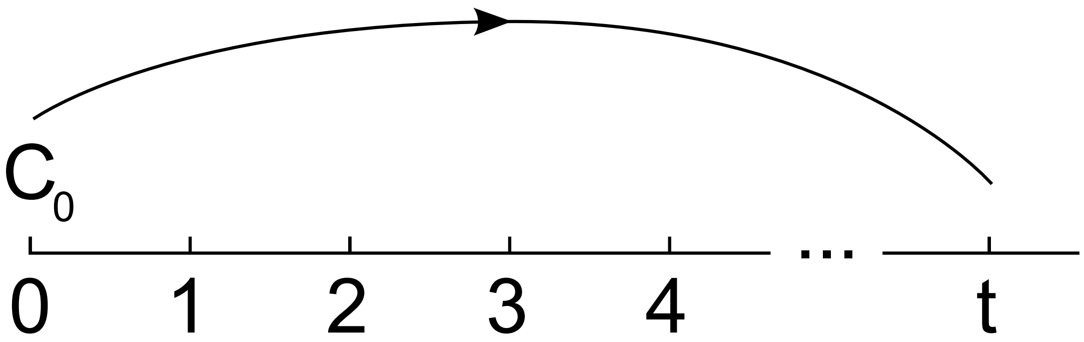
$$\boldsymbol{FV(single\ cash\ flow\ ) = V_t = C_0(1+r)^t}$$

Where:

C_0 = cash flow now.

t = time periods into the future.

r = the effective rate over a single period.



Calculation Example: <u>Future</u> Value of a Single Cash Flow

Question: You have \$100 in the bank. Interest rates are 8% pa. How much will you have in the bank after 5 years?

Answer:

$$
\begin{aligned}
V_t &= C_0(1+r)^t \\
&= 100 \times (1+0.08)^5 \\
&= 146.9328
\end{aligned}
$$

Calculation Example: Present and Future Values

Question: If you pay this year's university fees of $5,000 now, the government will give you a 25% discount. Otherwise the government will lend you the $5,000, but will capitalise interest charges (add interest charges to the principle) at the rate of inflation which is expected to be 2.5% pa, compounding annually. When you start work, the government will demand repayment of your debt. You expect to start work and have to pay off all of your debt in a single payment in 6 years.

You can borrow and lend at 8% pa from and to the bank.

Should you pay your fees now or in 6 years?

Answer:

Option 1: Pay your uni fees now:

If you pay your fees now and receive the 25% discount, they will cost:

$$V_0 = 5{,}000 \times (1 - 0.25$$

$$= \mathbf{3,750}$$

Option 2: Pay your university fees in 6 years:

The future value of the uni fees in 6 years, growing at the inflation rate of 2.5% pa will be:

$$V_t = C_0(1 + r)^t$$

$$V_6 = 5{,}000 \times (1 + 0.025)^6 = \mathbf{5,798.4671}$$

This is the amount that the government will demand for repayment. A naive person would compare this to the $3,750 and conclude that paying immediately is better, but that is wrong. The two amounts can't be compared since they are at different times. Values can only be compared at the same point in time.

The present value of $5,798.47 can be calculated and compared to the cost of paying the fees now.

$$V_0(1+r)^t = V_6$$

$$V_0(1+0.08)^6 = 5{,}798.4671$$

$$V_0 = \frac{5{,}798.4671}{(1+0.08)^6} = \mathbf{3{,}654.0178}$$

Therefore you should pay your fees in 6 years since it has a lower present value of costs.

Present Value of an Annuity

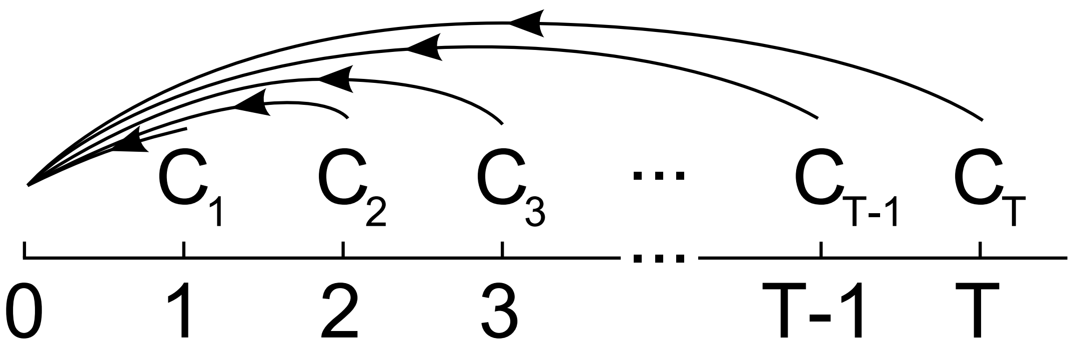
$$\boldsymbol{PV(annuity) = V_0 = \frac{C_1}{r}\left(1 - \frac{1}{(1+r)^T}\right)}$$

Where:

C_1 = the cash flow received at $t = 1$ and every period after until the last cash flow at $t = T$. All cash flows are equal to C_1, they don't grow.

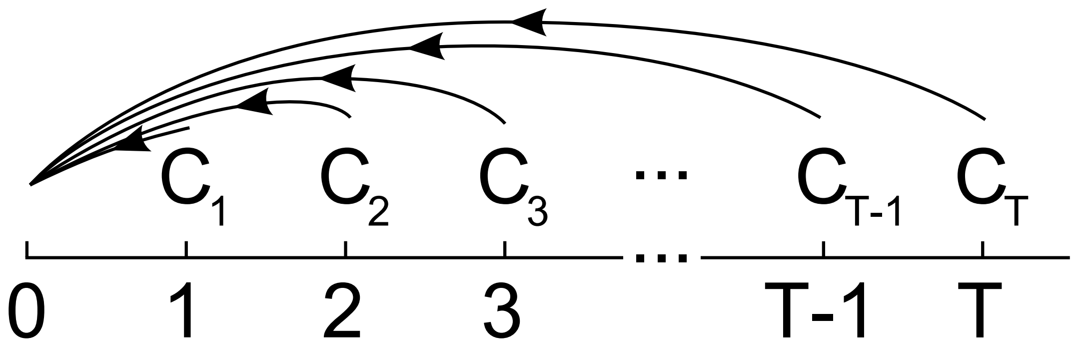
T = time periods until the last cash flow.

r = the effective rate over a single period.



$$PV(annuity) = V_0 = \frac{C_1}{r}\left(1 - \frac{1}{(1+r)^T}\right)$$

Note that C_1 is used instead of C to remind you that the first cash flow is 1 period ahead of the present value V_0. The annuity formula does not include a cash flow at $t = 0$.



Calculation Example: Present Value of an Annuity

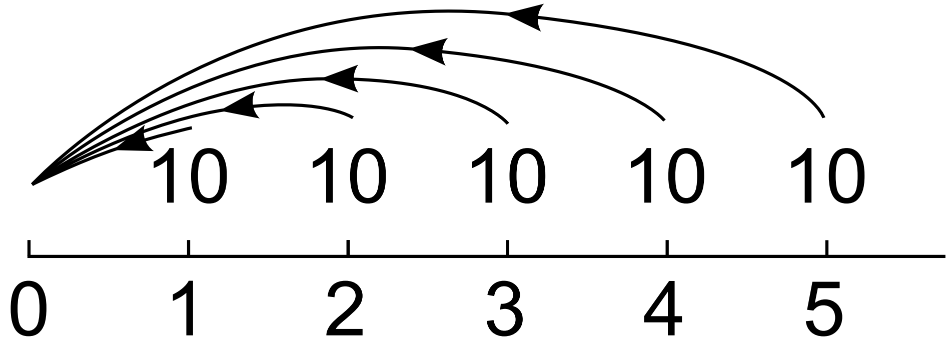
Question: What is the value of receiving \$10 for the next 5 years with the first payment one year from now? The interest rate is 10% pa.

Answer:

$$V_0 = \frac{C_1}{r}\left(1 - \frac{1}{(1+r)^T}\right)$$

$$= \frac{10}{0.1}\left(1 - \frac{1}{(1+0.1)^5}\right)$$

$$= 37.9079$$



Present Value of a Perpetuity with Growth

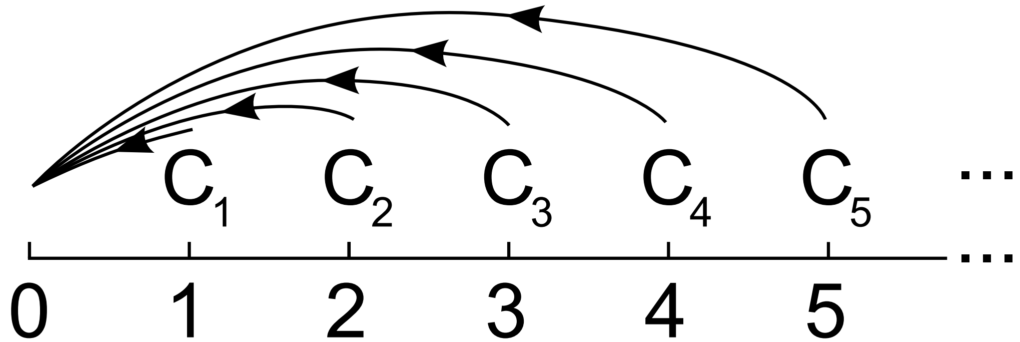
$$PV(perpetuity\ with\ growth) = V_0 = \frac{C_1}{r - g}$$

Where:

C_1 = cash flow received at $t = 1$. The cash flows go on forever, but grow by g every period..

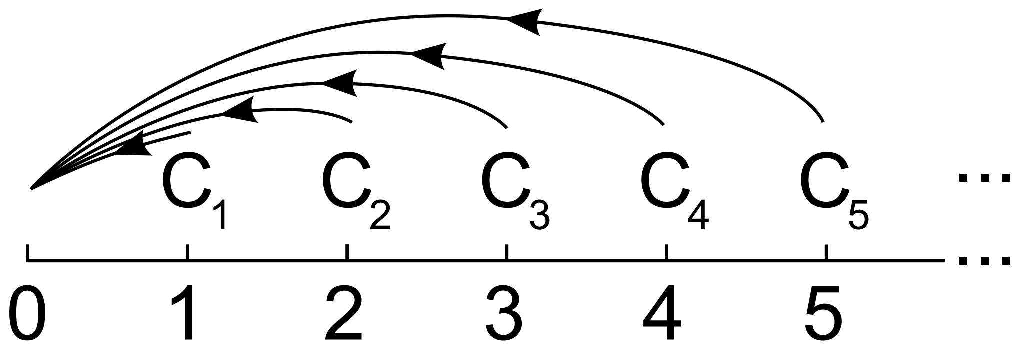
g = effective growth rate over a single period.

r = effective discount rate over a single period.



Note that C_1 is used instead of C to remind you that the first cash flow is 1 period ahead of the present value V_0. The perpetuity formula does not include a cash flow at $t = 0$.

Note that:



$C_2 = C_1(1 + g)$

$C_3 = C_1(1 + g)^2$

$C_4 = C_1(1 + g)^3$ and so on.

Perpetuities with no growth are called level perpetuities. In this case, $g = 0$ and $C_1 = C_2 = C_3$ and so on.

Calculation Example: Present Value of a Perpetuity (with no growth)

Question: Your friend promises to pay you \$50 every year forever, if you lend him \$400 now. Interest rates are 10% pa. Is this a good deal for you?

Answer: Your friend is offering you a perpetuity with no growth.

$$V_0 = \frac{C_1}{r - g}$$

$$= \frac{50}{0.1 - 0}$$

$$= 500, \text{ which is more than \$400 so it is a good deal.}$$

But this is assuming that your friend actually does pay you forever even after he becomes old and senile. Assuming he stops paying you in 40 years, then the $50 will be an annuity:

$$V_0 = \frac{C_1}{r}\left(1 - \frac{1}{(1+r)^T}\right)$$

$$= \frac{50}{0.1}\left(1 - \frac{1}{(1+0.1)^{40}}\right)$$

$= 488.9525$, which is more than $400, so it's still a good deal.

Annualised Percentage Rates (APR's)

Most interest rates are quoted as Annualised Percentage Rates (APR's). This is true for credit card rates, mortgage rates, bond yields, and many others. APR's are sometimes called nominal rates, but nominal has another meaning related to inflation so we will avoid calling APR's nominal rates.

The compounding period of an APR is usually not explicitly stated. But usually it can be assumed that the compounding frequency is the same as the payment frequency.

For example, a credit card might advertise an interest rate of 24% pa. This is an APR. Because credit cards are always paid off monthly, the compounding frequency is per month. Therefore the interest rate is 24% pa compounding monthly.

Effective Rates

Effective rates compound only once over their time period, and the time period can be of any length, not necessarily annual.

Effective rates can be used to discount cash flows.

APR's **cannot** be used to discount cash flows, they must be converted to effective rates first.

Note that all of the calculation examples up to here have assumed that the interest rate given is an effective rate.

Calculation Example: Present Values and Effective Rates

Question: What is the present value of receiving $100 in one year when the effective monthly rate is 1%?

Answer: Since the effective interest rate is per month, the time period must also be in months, so

$$V_o = \frac{C_t}{(1+r)^t}$$

$$= \frac{100}{(1+0.01)^{12}}$$

$$= 88.7449$$

APR's and Effective Rates

- An APR compounding monthly is equal to 12 multiplied by the effective monthly rate.

$$r_{APR,comp\ monthly} = r_{eff,monthly} \times 12$$

- An APR compounding semi-annually is equal to 2 multiplied by the effective 6 month rate.

$$r_{APR,comp\ per\ 6mths} = r_{eff,6mth} \times 2$$

- An APR compounding daily is equal to 365 multiplied by the effective daily rate.

$$r_{APR,comp\ daily} = r_{eff,daily} \times 365$$

Calculation Example: Future Values with APR's

Question: How much will your credit card debt be in 1 year if it's $1,000 now and the interest rate is 24% pa?

Answer: Since credit cards are paid off per month, the 24% must be an APR compounding monthly. Therefore the effective monthly rate will be the APR divided by 12.

$$r_{eff,monthly} = \frac{r_{APR,comp\ monthly}}{12} = \frac{0.24}{12} = 0.02$$

$$V_t = C_0(1+r)^t$$

$$= 1000(1+0.02)^{12} = 1{,}268.2418$$

Converting Effective Rates To Different Time Periods

Compounding the rate higher (to a longer time period):

$$r_{eff,annual} = \left(1 + r_{eff,monthly}\right)^{12} - 1$$

$$r_{eff,semi-annual} = \left(1 + r_{eff,monthly}\right)^{6} - 1$$

$$r_{eff,quarterly} = \left(1 + r_{eff,monthly}\right)^{3} - 1$$

Compounding the rate lower (to a shorter time period):

$$r_{eff,monthly} = \left(1 + r_{eff,annual}\right)^{\frac{1}{12}} - 1$$

$$r_{eff,daily} = \left(1 + r_{eff,annual}\right)^{\frac{1}{365}} - 1$$

Calculation Example: Converting Effective Rates

Question: A stock was bought for $10 and sold for $15 after 7 months. No dividends were paid. What was the effective annual rate of return?

Answer:

First we find the return over 7 months. This will be the effective 7 month rate of return. Note that the time period is in 7-month blocks, so t=1:

$$V_o = \frac{V_t}{(1+r)^t}$$

$$V_o = \frac{V_1}{(1+r)^1}$$

$$10 = \frac{15}{(1+r)^1}$$

$$(1+r)^1 = \frac{15}{10}$$

$r = \frac{15}{10} - 1 = 0.5 = 50\%$, which is the effective 7 month rate.

Now we need to convert it to an effective annual rate. This can be done in one step as follows:

$$r_{eff,annual} = \left(1 + r_{eff,7mth}\right)^{\frac{12}{7}} - 1$$

$$= (1 + 0.5)^{12/7} - 1 = 1.0039 = 100.39\%$$

Or it can be broken down into two steps:

- Compounding the 7-month rate down to a monthly rate:

$$r_{eff,monthly} = \left(1 + r_{eff,7mth}\right)^{1/7} - 1$$
$$= (1 + 0.5)^{1/7} - 1 = 0.059634 = 5.9634\%$$

- Then compound the monthly rate up to a 12-month (annual) rate:

$$r_{eff,annual} = \left(1 + r_{eff,monthly}\right)^{12} - 1$$
$$= (1 + 0.059634)^{12} - 1 = 1.0039 = 100.39\%$$

Calculation Example: Converting APR's to Effective Rates

Question: You owe a lot of money on your credit card. Your credit card charges you 9.8% pa, given as an APR compounding per month.

You have the cash to pay it off, but your friend wants to borrow money from you and offers to pay you an interest rate of 10% pa given as an effective annual rate.

Should you use your cash to pay off your credit card or lend it to your friend?

Assume that your friend will definitely pay you back (no credit risk).

Answer:

Since the loan interest rate is an effective rate but the credit card rate is an APR we can't compare them.

Let's convert the credit card's 9.8% APR compounding per month to an effective annual rate:

$$r_{eff,monthly} = \frac{r_{APR,comp\ monthly}}{12}$$

$$= \frac{0.098}{12} = 0.0081667$$

$$r_{eff,annual} = \left(1 + r_{eff,monthly}\right)^{12} - 1$$

$$= (1 + 0.0081667)^{12} - 1 = 0.1025 = 10.25\%$$

So the credit card's 9.8% APR compounding per month converts to an effective annual rate of 10.25%. This is more than the loan's 10% effective annual rate, so you should pay off your credit card instead of lending to your friend.

Calculation Example: Home Loans

Question 1: Fully amortising mortgages

Mortgage rates are currently 6% and are not expected to change.

You can afford to pay $2,000 a month on a mortgage.

The mortgage term is 30 years (matures in 30 years).

No deposit is required.

What is the most expensive house you can buy if your mortgage is fully amortising? Fully amortising means that the mortgage will be fully paid off at maturity (in 30 years).

Answer: Since the mortgage is fully amortising, at the end of the loan's maturity the whole loan will be paid off.

The bank will lend you the present value of your monthly payments for the next 30 years. This can be calculated using the annuity formula.

The $2,000 payments are monthly, therefore the interest rate and time periods need to be measured in months too.

$$t = 30 \times 12 = 360\ months$$

$$r_{eff,monthly} = \frac{r_{APR,comp\ monthly}}{12}$$

$$= \frac{0.06}{12} = 0.005 = 0.5\%$$

$$
\begin{aligned}
V_0 &= \frac{C_1}{r}\left(1 - \frac{1}{(1+r)^T}\right) \\
&= \frac{2000}{0.005}\left(1 - \frac{1}{(1+0.005)^{360}}\right) \\
&= \$333{,}583
\end{aligned}
$$

Question 2: Interest-only mortgages

What is the most expensive house you can buy if your mortgage is interest only? 'Interest only' means that all of your payments go towards the interest, so the principal is never paid off. At the end of the 30 year mortgage you will have to pay the original principal, usually by re-borrowing/re-financing.

Answer: Very similar to the last question, but the present value also includes the big payment at the end which is equal to the original amount we are borrowing (V_0):

$$V_0 = \frac{C_1}{r}\left(1 - \frac{1}{(1+r)^T}\right) + \frac{V_0}{(1+r)^T}$$

$$V_0 = \frac{C_1}{r}\left(1 - \frac{1}{(1+r)^T}\right) + \frac{V_0}{(1+r)^T}$$

$$V_0 - \frac{V_0}{(1+r)^T} = \frac{C_1}{r}\left(1 - \frac{1}{(1+r)^T}\right)$$

$$V_0\left(1 - \frac{1}{(1+r)^T}\right) = \frac{C_1}{r}\left(1 - \frac{1}{(1+r)^T}\right)$$

$$V_0 = \frac{C_1}{r}$$

$$= \frac{2000}{0.005}$$

$$= \$400{,}000$$

Calculation Example: Solving for Time

Question 1: You have some money in the bank. Interest rates are 6% pa. How long will it take before your money in the bank has doubled?

Answer: Let x be the money currently in the bank. Therefore the money in the bank will double when the future value is $2x$. Assuming that the 6% interest rate is an APR compounding per month, then the effective monthly rate will be 0.005 or 0.5%.

$$V_t = V_0(1+r)^t$$

$$2x = x\left(1+\frac{0.06}{12}\right)^t$$

$$2 = (1 + 0.005)^t$$

$$1.005^t = 2$$

$$\log(1.005^t) = \log(2)$$

$$t \times \log(1.005) = \log(2)$$

$$t = \frac{\log(2)}{\log(1.005)} = 138.98 \text{ months} = 11.58 \text{ years.}$$

Question 2: You want to borrow $500,000 now to buy a house. You can afford to pay $4,000 per month towards the mortgage. The interest rate on the mortgage is 9% pa. How long will it take you to pay it off?

Answer: The mortgage is fully amortising since the payments must completely pay off the loan at maturity. So,

$$V_0 = \frac{C_1}{r}\left(1 - \frac{1}{(1+r)^T}\right)$$

$$500{,}000 = \frac{4{,}000}{(0.09/12)}\left(1 - \frac{1}{\left(1 + (0.09/12)\right)^T}\right)$$

$$\frac{500{,}000}{4{,}000} = \frac{1}{(0.0075)}\left(1 - \frac{1}{(1 + 0.0075)^T}\right)$$

$$\frac{500 \times 0.0075}{4} = 1 - \frac{1}{1.0075^T}$$

$$0.9375 = 1 - \frac{1}{1.0075^T}$$

$$0.0625 = \frac{1}{1.0075^T}$$

$$1.0075^T = \frac{1}{0.0625}$$

$$= 16$$

$\log(1.0075^T) = \log(16)$, and using log rules,

$$t \times \log(1.0075) = \log(16)$$

$$t = \frac{\log(16)}{\log(1.0075)} = 371.06 \text{ months} = 30.92 \text{ years}$$

Calculation Examples: Discounting Cash Flows

Question 1: You will receive 4 payments of $100. The first payment is in 7 years, the second in 8 years, and so on until the 10th year. Interest rates are 6% pa, given as an APR compounding monthly. What is the present value of these payments?

Answer:

We will use the annuity equation for the four $100 cash flows. What makes it a little more difficult is that the annuity payments do not start at t=1. They start at t=7 so the value produced by the annuity equation will actually be 1 period before the first cash flow which is t=6. The effective annual interest rate will be $r_{eff,annual} = \left(1 + \frac{0.06}{12}\right)^{12} - 1 = 0.061678$

Value of the four $100 cash flows:

$$V_0 = \frac{C_1}{r}\left(1 - \frac{1}{(1+r)^T}\right)$$

$$V_6 = \frac{100}{0.061678}\left(1 - \frac{1}{(1+0.061678)^4}\right)$$

$$= 345.1832, \text{ but this is a value in the 6th year.}$$

$$V_0 = \frac{V_t}{(1+r)^t} = \frac{V_6}{(1+0.061678)^6} = \frac{345.1832}{(1+0.061678)^6}$$

$$= \ 241.04$$

Question 2: You will receive a payment of $100 now and every year in the future forever, growing at an effective rate of 5% pa. So the payment at the end of the first year (t=1) will be $105, and so on. Interest rates are 6% pa, given as an APR compounding monthly. What is the present value of these payments?

Answer:

Since the cash flows are received per year, the time period must be yearly and the discount rate and growth rate must be effective annual rates. The growth rate of 5% is already an effective annual rate. But the 6% interest rate is an APR compounding per month which needs to be converted to an effective annual rate:

In one step:

$$r_{eff,annual} = \left(1 + \frac{r_{APR,comp\ monthly}}{12}\right)^{12} - 1$$

$$= \left(1 + \frac{0.06}{12}\right)^{12} - 1 = 0.0616778$$

Or in two steps:

$$r_{eff,monthly} = \frac{r_{APR,comp\ monthly}}{12}$$

$$= \frac{0.06}{12} = 0.005$$

$$r_{eff,annual} = \left(1 + r_{eff,monthly}\right)^{12} - 1$$

$$= (1 + 0.005)^{12} - 1 = 0.0616778$$

Since the cash flows go on forever, we will use the perpetuity formula. But we must be careful since the perpetuity formula assumes that the first cash flow (C_1) is at t=1, not t=0 as is the case in this question. Therefore $C_1 = \$105$, which will be input into the perpetuity formula, and the $100 received at t=0 will simply be added on since it doesn't need discounting because it is received at t=0.

$$V_{0,perp} = \frac{C_1}{r - g}$$

$$= \frac{105}{0.0616778 - 0.05} = 8{,}991.41$$

$$V_{0,total} = V_{0,perp} + 100$$

$$= 8{,}991.41 + 100 = \$9{,}091.41$$

Calculation Examples: Converting Rates

A 10 year bond has a yield of 5%. It pays semi-annual coupons.

Q1) Find the effective 6 month rate.

Since the bond pays semi-annual coupons (as is customary for all bonds in Australia and the US), the yield of 5% must be quoted as an APR with semi-annual (6 month) compounding.

$$r_{eff,6mth} = \frac{r_{APR,comp\ semi-annually}}{2}$$

$$= \frac{0.05}{2} = 0.025$$

Q2) Find the effective annual rate.

$$r_{\text{eff,annual}} = \left(1 + \frac{r_{\text{APR,comp semi-annually}}}{2}\right)^2 - 1$$

$$= \left(1 + \frac{0.05}{2}\right)^2 - 1 = 0.050625 = 5.0625\%$$

Q3) Find the effective monthly rate.

$$r_{\text{eff,monthly}} = \left(1 + \frac{r_{\text{APR,comp semi-annually}}}{2}\right)^{1/6} - 1$$

$$= \left(1 + \frac{0.05}{2}\right)^{1/6} - 1 = 0.004123915 = 0.4124\%$$

Q4) Find the APR compounding every month.

$$r_{\text{APR,comp monthly}} = r_{\text{eff,monthly}} \times 12$$

$$= \left[\left(1 + \frac{r_{\text{APR,comp semi-annually}}}{2}\right)^{1/6} - 1\right] \times 12$$

$$= \left[\left(1 + \frac{0.05}{2}\right)^{1/6} - 1\right] \times 12$$

$$= 0.004123915 \times 12$$

$$= 0.049486986 = 4.949\%$$

Q5) What is the present value of $100 received in 1 year, using the 5% bond yield as the discount rate?

The 5% APR compounding every 6 months can't be used to discount anything, it must be converted to an effective rate of some sort, which we've already done.

Let's use the effective 6 month rate, $r_{eff,6mth} = 0.025$. Since the rate is in 6-month periods, then the time must also be in 6-month periods. The cash flow is one year away so t=2 six-month periods.

$$V_0 = \frac{C_t}{(1+r)^t}$$

$$V_0 = \frac{100}{(1+0.025)^2} = 95.1814$$

Using the effective monthly rate ($r_{eff,monthly} = 0.004123915$), with t=12 months, then we get the same answer:

$$V_0 = \frac{C_t}{(1+r)^t}$$

$$V_0 = \frac{100}{(1+0.004123915)^{12}} = 95.1814$$

Inflation and Rates of Return

Inflation is the increase in the general level of prices in an economy. Positive inflation reduces the buying power of money.

Returns (effective returns or APR's) are usually stated as 'nominal' rates which means that they have not been reduced by the rate of inflation. To turn a nominal return into a real return, we use the Fisher equation:

$$1 + r_{real\ return} = \frac{1 + r_{nominal\ return}}{1 + r_{inflation}}$$

Note that the rates used in the above equation should be effective rates, not APR's.

Confusion: The Term 'Nominal' is Ambiguous

Be aware that Annualised Percentage Rates (APR's) are also sometimes called 'nominal rates' even though they have nothing to do with the concept of inflation.

This is very confusing. In these notes, when a 'nominal rate' is mentioned, it means a rate that is not adjusted for inflation.

Unfortunately many textbooks do not specify which definition of 'nominal' they are using.

Calculation Example: Inflation and Returns

Question: Fred bought an investment property for $500,000. He sold it one year later for $550,000. Over that year the Consumer Price Index (CPI) rose from 110 to 123.2. What was the nominal return and real return on the investment property? Assume that nothing was spent on the property and it wasn't rented, so the only cash flows are the buy and sell amounts.

Answer: First let's find the nominal return on the property:

$$V_0 = \frac{V_1}{(1+r)^1}$$

$$500{,}000 = \frac{550{,}000}{(1+r)^1}$$

$$r = \frac{550{,}000}{500{,}000} - 1 = 0.1 = 10\%$$

This is an effective annual rate, and it is also a **nominal** rate of return since it hasn't been reduced by inflation. To find the real rate of return, we can find the rate of inflation then use the Fisher equation.

$$V_0 = \frac{V_1}{(1+r)^1}$$

$$110 = \frac{123.2}{\left(1 + r_{inflation}\right)^1}$$

$$r_{inflation} = \frac{123.2}{110} - 1 = 0.12$$

which is an effective annual rate.

Since the nominal return and inflation are both effective annual rates we can use them in the Fisher equation to find the real return:

$$1 + r_{real\ return} = \frac{1 + r_{nominal\ return}}{1 + r_{inflation}}$$

$$1 + r_{real\ return} = \frac{1 + 0.1}{1 + 0.12}$$

$$r_{real\ return} = \frac{1 + 0.1}{1 + 0.12} - 1$$

$$= -0.017857143$$

$$= -1.179\%$$

So the investment property was actually not a great investment since its real rate of return was negative. However, if Fred didn't buy the property and just kept his cash under his bed, he would have had a real return of -12% (the rate of inflation).

Note that the real rate of return (-1.179%) is ***not*** equal to the nominal rate minus inflation (10% - 12%). You have to use the Fisher equation to calculate the exact real rate of return.